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Torsional Stiffness of Interference Fit Connections

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A new method to determine the torsional stiffness of an interference fit shaft connection is established. This method is intended to replace the empirical rule used to date, by which the connection's stiffness is only a function of the hub bore and length. The mathematical analysis shows that the torsional stiffness of an interference fit connection is not constant, as generally thought, but it is a function of many factors, including the transmitted torque and the rotating speed. The validity of the method was confirmed through a large number of laboratory tests.

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Torsional Stiffness of Interference Fit Connections

M. M. CALISTRAT G. G. LEASEBURGE

INTRODUCTION

Although the amount of literature available on the subject of torsional vibrations is quite large, little can be found about the stiffness of a shaft to hub connection. Ker Wilson (1)[1] established an empirical method by which the shaft is regarded as unrestrained by the hub for a distance of one-third the length of the hub. This "one-third" rule is applied whether the connection is keyed, splined, or interference fitted. This method seemed satisfactory, especially in cases where the stiffness of the connection was high in comparison with the rest of the system. Ker Wilson cautions about loosely fitted hubs and about the tendency for the interference fit to loosen due to expansion of the bore caused by centrifugal stress, but the influence of these factors is not analyzed.

The ever-increasing use of high-speed machinery changes somewhat the picture not only because of the increase in speed, but also because the torsional stiffness of the devices used to connect two shafts is now constituting a substantial portion of the system's stiffness.

Two areas are of special interest: (a) the shaft to hub connection and (b) the flange to flange connection. This paper analyzes these two areas and formulas for calculation of the connections' stiffness are given. A large amount of experimental work was done to establish the reliability of the formulas and the necessary coefficients.

FLANGED HUB (SHAFT CONNECTION)

Fig. 1 shows one of the most common means of transmitting the torque from one shaft to another. In a keyless connection, the torque can be transmitted only if the hub is mounted on the shaft with an interference fit, and the maximum torque that can be transmitted is a function of the amount of shaft to hub interference fit, the dimensions of the hub, and the friction coefficient at the interface.

[1] Underlined numbers in parentheses designate References at end of paper.

NOMENCLATURE

- θ = angular deflection
- T = torque
- L = axial length, measured from the plane where the shaft penetrates the hub
- P = contact pressure between hub and shaft
- f = friction coefficient
- r = radius of shaft, in the plane A-A
- r_o = radius of shaft at the hub entrance, for tapered shafts
- R = outside radius of hub
- G = modulus of rigidity
- J = polar moment of inertia
- E = modulus of elasticity
- i = interference ratio (radial interference/shaft radius)
- ν = material density
- μ = Poisson ratio
- ω = angular velocity
- g = acceleration due to gravity
- e = deflection in the flange bolts
- B = bolt circle diameter
- F = tangential force on flange bolts
- N = number of flange bolts
- d = diameter of flange bolts
- K = torsional stiffness of flange connection
- m = slope of tapered bore (radius variation per axial unit length)
- k = correction factor for bolted flanges

Subscripts

- s = pertaining to shaft
- h = pertaining to hub
- i = pertaining to interference
- c = pertaining to centrifugal effect
- p = pertaining to penetration

In the plane where the shaft enters the hub, the torque in the shaft, T_s, is equal to the total torque, "T." Over an infinitesimal length, "dL," the angular deflection of the shaft per unit length is:

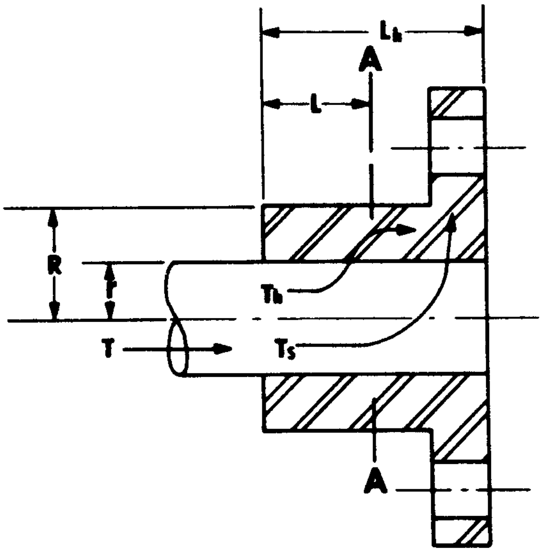

Fig. 1

$$\frac{d\theta_s}{dL} = \frac{T_s}{G_s \times J_s} \quad (1)$$

The following conditions apply in a plane A-A placed at the distance, "L," from the plane where the shaft enters the hub.

Part of the total torque, "T," is transmitted to the hub through friction from L = 0 to L = L. This torque is expressed as:

$$T_h = \int_o^L 2\pi\ f\ P\ r^2\ dL \quad (2)$$

The torque in the shaft, in the plane A-A, is obtained from:

$$T_s = T - T_h \quad (3)$$

It can be seen that while the torque in the hub increases as the plane A-A penetrates inside the assembly, the torque in the shaft decreases. The angular deflection per unit length is directly proportional to the torque. The angular deflection per unit length of the shaft is, hence, not constant, but decreases as L increases. The reverse is true for the hub. It is likely that at a given distance, "L_p," the angular deflection per unit length of the shaft becomes equal to the angular deflection per unit length of the hub.

$$\frac{d\ \theta_s}{dL} = \frac{d\ \theta_h}{dL} = \frac{T_s}{G_s \times J_s} = \frac{T_h}{G_h \times J_h} \quad (4)$$

Equal angular deflections per unit length mean no

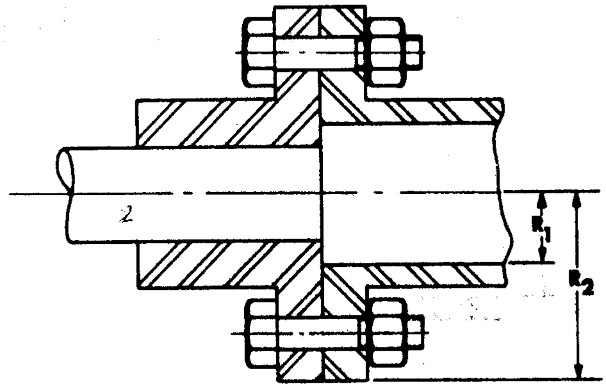

Fig. 2

relative motion between shaft and hub. Beyond the plane where relative motion stops, the shaft-hub assembly can be considered a unitized body.

The length, "L_p," to the plane where relative motion stops is called "penetration." From equations (2) and (4), the value of the penetration can be determined. To simplify the calculations, it is assumed that the hub and shaft are made out of the same material. Formula (4) becomes:

$$\frac{T_s}{r^4} = \frac{T_h}{R^4 - r^4} \quad (5)$$

Solving equation (2) for L_p

$$T_h = 2\pi\ f\ P\ r^2\ L_p \quad (6)$$

$$T_s = 2\pi\ f\ P\ L_p\ r^6/(R^4 - r^4) \quad (7)$$

$$L_p = \frac{T\ (R^4 - r^4)}{2\pi\ f\ P\ r^2\ R^4} \quad (8)$$

It should be noted that the penetration is a function of the transmitted torque.

The angular deflection of the shaft-hub assembly can be calculated by summing the deflection of the shaft for the length of penetration and the deflection of the remainder of the assembly under the total torque. The deflection of the shaft inside the hub is:

$$\theta_s = \int_o^{L_p} \frac{T_s\ dL}{G \times J_s} \quad (9)$$

$$\theta_s = \frac{2\ T\ L_p}{G\ \pi\ r^4} - \frac{2\ f\ P\ L_p^2}{G\ r^2} \quad (10)$$

The angular deflection of the remainder of the assembly can be easily calculated (4).

The contact pressure can be calculated as the sum of the pressure generated by the interference fit and the pressure generated by the centrifugal forces.

$$P = P_i + P_c \qquad (11)$$

$$P_i = \frac{Ei\ (R^2 - r^2)}{2\ R^2} \qquad (12)$$

$$P_c = -\frac{\upsilon}{g} \times \frac{\omega^2}{8}(3 + \mu)\ (R^2 - r^2) \qquad (13)$$

From the foregoing formulas, it can be seen that the stiffness of a shaft to hub connection is influenced by:

1 The torque transmitted: θ_s is a function of the square of the torque, the deflection of the remainder of the assembly is a function of the penetration, i.e., of the torque.

2 The ratio between the shaft and hub radii: Both r and R enter in the calculation of the penetration.

3 The interference between hub and shaft: It influences P.

4 The friction coefficient at the interface.

5 The rotating speed: The interference between hub and shaft decreases with the increase in centrifugal forces.

The influcence of speed on the torsional stiffness becomes significant only at very high speeds or at reduced interferences.

The flange geometry does not normally enter in these calculations. It was shown that the contact pressure influences the stiffness only for the length of penetration. The penetration cannot be larger than the cylindrical body of the hub, because total slippage occurs before that.

Although penetration is defined as the length of hub at which motion between shaft and hub stops, slippage will occur before the penetration equals the total length, "L_h." To comprehend this, one must realize that the torque in the shaft (T_s) in the plane where relative motion stops is transmitted to the hub also by friction; thus a minimum remaining length of contact must exist. The torque at which the shaft slips inside the hub is determined by:

$$T_{max.} = 2\ \pi\ f\ P\ r^2\ L_h \qquad (14)$$

The friction coefficient, "f," must be determined experimentally, for the particular materials and surface qualities used.

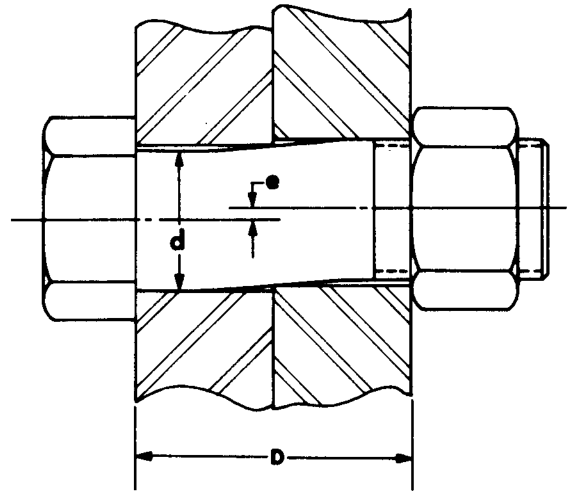

Fig.3

FLANGED HUB (FLANGE CONNECTION)

Fig. 2 shows a typical flange connection. The torque that can be transmitted from one flange to the other through friction is very small. This type of connection transmits the torque through the bolts. For as long as the torque can be transmitted through friction, the connection is torsionally very stiff. Once the torque at which the flanges could slip is exceeded, the connection becomes much softer, but the torsional stiffness is independent of torque. One should not neglect considering the stiffness of the flange connections; it was found to represent a substantial percentage of the system's stiffness.

The "slip-torque" for a flange connection is a function of the dimensions of the flanges, the size and number of bolts, and the tightening torque of the bolts. The method to calculate the "slip-torque" is found in most machine design manuals, such as the one by Black (2).

Once the slip-torque is exceeded, as is the case for most applications, the torque is transmitted through the bolts. Fig. 3 represents the bolt connection, under torque.

The following notations are used to determine the stiffness of the flange connection.

$$K = \frac{\text{Torque}}{\text{Angular deflection}} = \frac{B \times N \times F/2}{2\ e\ /\ B} \qquad (15)$$

$$e = \frac{D \times F}{G \times \pi \times d^2/4} \qquad (16)$$

$$K = \frac{B^2 \times N \times G \times \pi \times d^2}{16 \times D} \qquad (17)$$

It was found experimentally that not all the bolts

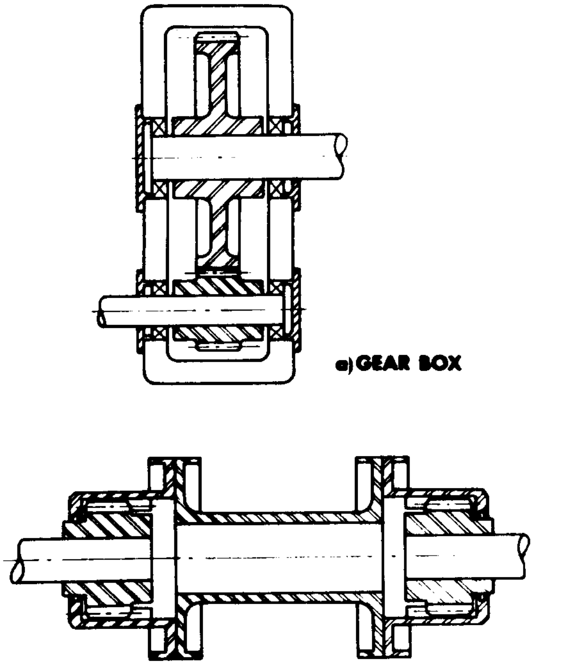

Fig.4

transmit the torque, and an experimental correction factor must be applied.

$$K = \frac{B^2 \times N \times G \times \pi \times d^2}{16 \times D \times k} \qquad (18)$$

GEARED HUB

Fig. 4 represents two applications where the torque is transmitted from the shaft to a gear. In both cases, the formulas established previously can be used, but a new condition can appear because the torque is no longer transmitted through a flange at the end of the hub. Such a condition is shown in Fig. 5.

It is possible that the penetration goes beyond the plane where the torque leaves the hub. In such a case, the torque from the shaft, T_s, goes first to the right of section A-A, but must return to the left of the section to be transmitted to the teeth.

Because the mechanism of torque transmittal remains the same, the previously established formulas for determining T_s and penetration are still valid. The total angular deflection from the plane where the shaft enters the hub to the plane of the teeth is, however, different.

The angular deflection in the shaft remains, θ_s, as shown in equation (10). To this, the deflection in the hub only, from the plane A-A to the teeth, must be added. The deflection in the hub is caused by two torques: T_{h2}, which varies from zero in the plane of the teeth to a maximum in plane A-A; and T_s, which is constant.

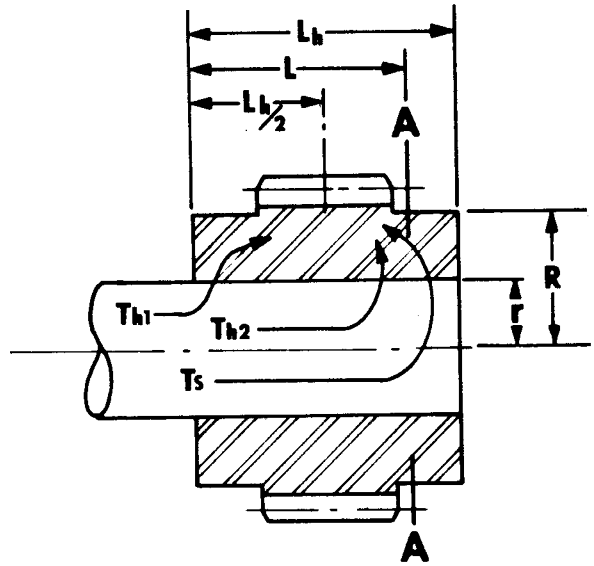

Fig.5

The angular deflection in the hub can be expressed as:

$$\theta_h = \int_{L_h/2}^{L_P} \frac{T_{h2}\, dL}{G J_h} + \frac{T_s \left(\frac{L_h}{2} - L_P\right)}{G J_h} \qquad (19)$$

T_{h2} is calculated using equation (2), and T_s was calculated in equation (7).

$$\theta_h = \frac{2\,T\,(L_P - L_h/2)}{G\,\pi\,(R^4 - r^4)} - \frac{2\,f\,P\,r^2\,(L_P - L_h/2)}{G\,(R^4 - r^4)} \qquad (20)$$

Another factor, specific only to the geared hub, is the stiffness of the teeth. When only a few teeth transmit the torque, as shown in Fig. 4(a), the deflection of the teeth can be proportionally high and should be considered. When all the teeth are loaded, as shown in Fig. 4(b), then the added angular deflection is minimal and can be neglected. The method to calculate the tooth deflection can be found in many gear manuals, such as the one by Buckingham (3).

TAPERED BORES

Although the amount of taper used in shaft connection is generally very small, it cannot be neglected, because it influences the torsional stiffness in more than one way:

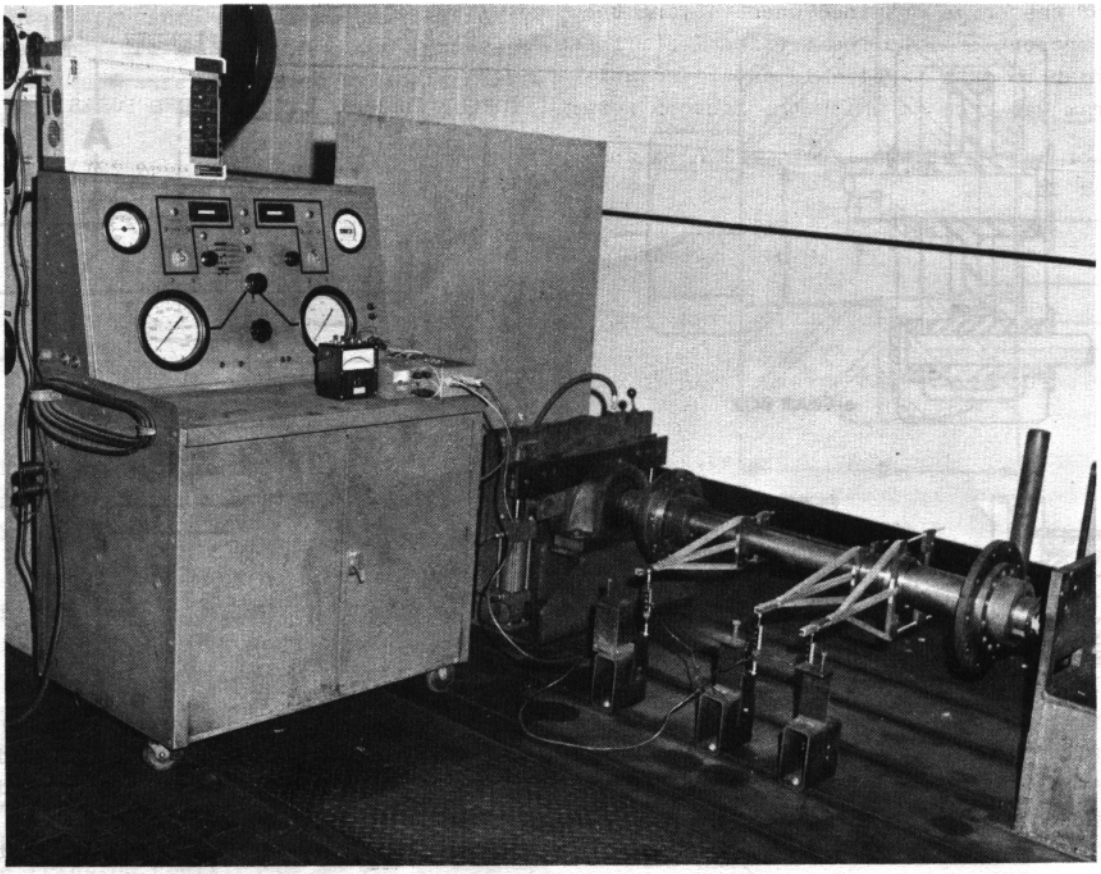

Fig.6

1 The diameter of the shaft decreases making it softer.

2 The ratio between the hub to shaft diameters increases; thus, the contact pressure increases.

3 The hub centrifugal growth is smaller.

The mathematical interpretation of tapered bore connections becomes rather involved. It is recalled that the basic formula for the previous calculations was

$$T_h = \int_o^L 2\pi \; f \; P \; r^2 \; dL \qquad (2)$$

In solving it, the contact pressure was considered constant. For tapered bores, the radius of the shaft and the contact pressure are variable with L. If r_o is the bore radius at the large end, then the contact pressure is expressed as:

$$P = \frac{Ei \; r_o}{2r} - \frac{Ei \; r_o r}{2R^2} - \frac{\upsilon \omega^2 (3 + \mu) R^2}{8g} + \frac{\upsilon \omega^2 (3 + \mu) r^2}{8g} \qquad (21)$$

The solution for equation (2) when equation (21) is used, can be found in the Appendix. The angular deflection of the shaft is determined with the expression

$$\theta_s = \int_o^{L_P} \frac{T_s \; dL}{G \times J_s} \qquad (9)$$

The shaft's modulus of inertia is now variable with L. The solution for θ_s can be found in the Appendix.

EXPERIMENTS

The test rig is shown in Fig. 6. Basically, it is composed of a hydraulic torque applier and a rigid stand, between which a shaft assembly, incorporating hubs and flanges as per Figs. 2 or 4, was installed. Under torque, the shaft assembly twisted, and the amount of angular deflection was measured as linear displacement at the end of fixed length arms. The electric sensing elements controlled the movements of a pen in an X-Y graph recorder. The electrical signals were arranged so that the X displacement was proportional to the applied torque and the Y displacement to the relative radial displacement of two chosen planes.

The slope of the curve obtained when varying the torque was proportional to the torsional stiffness.

In order to obtain reliable data, a statistical program was used in which the factors were:

1 Type of hub — at two levels, flanged and geared
2 O.D. of hub — at three levels
3 Bore diameter — at two levels for each hub O.D.
4 Bore type — at two levels, straight and tapered
5 Interference fit — at three levels for each bore diameter.

A half factorial program was used.

The results of the tests not only confirmed the theory presented here, but also showed that the friction coefficient can be easily determined and that it is a function of materials and surface finish.

CONCLUSIONS

The main conclusion is that the torsional stiffness of a hub-to-shaft connection is not a constant. It is substantially influenced by the transmitted torque, the hub-to-shaft interference fit, and the ratio between the shaft and hub outside diameters. It is also influenced, to a lesser extent, by the rotating speed.

The torsional stiffness of a flanged connection can have two values depending upon the magnitude of the transmitted torque. At the torque levels considered normal, the connection's stiffness is a function of the dimensions and of the number of bolts, but independent of torque.

The mathematical formulas developed in this paper can be programmed in a computer so that the torsional stiffness of any particular connection can be obtained in a chart form as a function of torque and interference fit.

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APPENDIX

TORQUE IN HUBS WITH TAPERED BORES

From equations (2) and (21), it can be written

$$T_h = \int_0^L \left[K_1 (r_o - mL) - \frac{K_1}{R^2} (r_o - mL)^3 - K_2 R^2 (r_o - mL)^2 + K_2 (r_o - mL)^4 \right] dl \tag{22}$$

in which

$$K_1 = \pi f E i r_o \tag{23}$$

$$K_2 = \pi f \upsilon \omega^2 (3 + \mu)/4g \tag{24}$$

The solution for equation (22) is:

$$T_h = \left(\frac{K_2 m^4}{5}\right) L^5 + \left(\frac{K_1 m^4}{4R^2} - K_2 r_o m^3\right) L^4 + \left(2K_2 r_o m^2 - \frac{K_2 R^2 m^3}{3} - \frac{K_1 r_o m^2}{R^2}\right) L^3 + \left(K_2 R^2 r_o m - 2K_2 r_o^3 m + \frac{3K_1 r_o^2 m}{2R^2} - \frac{K_1 m}{2}\right) L^2 + \left(K_2 r_o^4 - K_2 R^2 r_o^2 - \frac{K_1 r_o^3}{R^2} + K_1 r_o\right) L \tag{25}$$

If each term in brackets is replaced by a constant

$$T_h = B_5 L^5 + B_4 L^4 + B_3 L^3 + B_2 L^2 + B_1 L \tag{26}$$

PENETRATION IN HUBS WITH TAPERED BORES

From equations (3) and (5)

$$T_h = T (1 - r^4/R^4) \text{ or} \tag{27}$$

$$T_h = T \; 1 - \left[(r_o - mL)^4/R^4\right] \tag{28}$$

Equalizing equations (26) and (28)

$$B_5 L^5 + \left(B_4 + \frac{Tm^4}{R^4}\right) L^4 + \left(B_3 - \frac{4Tr_o m^3}{R^4}\right) L^3 + \left(B_2 + \frac{6Tr_o^2 m^2}{R^4}\right) L^2 + \left(B_1 - \frac{4Tr_o^3 m}{R^4}\right) L + \frac{Tr_o^4}{R^4} - T = 0 \tag{29}$$

Again, the terms in brackets are replaced by constants

$$B_5 L^5 + C_4 L^4 + C_3 L^3 + C_2 L^2 + C_1 L + C_0 = 0 \tag{30}$$

The first positive real root of equation (30) yields the value for L_p

ANGULAR DEFLECTION OF TAPERED SHAFTS

The expression (9) can be used

$$\theta_s = \int_0^{L_p} \frac{T_s \, dL}{G \times J_s} \tag{9}$$

$$T_s = T - T_4$$

$$J_s = \frac{\pi}{2} (r_o - mL)^4 \tag{31}$$

T_h as defined in (26) (32)

The solution for equation (9) is

$$\theta_s = 2 \, (TA_1 + B_5A_6 - B_4A_5 - B_3A_4 - B_2A_3 - B_1A_2)/\pi G \tag{33}$$

in which

$$A_1 = \int_0^{L_p} \frac{dL}{(r_o - mL)^4} \tag{34}$$

$$A_2 = \int_0^{L_p} \frac{L \, dL}{(r_o - mL)^4} \tag{35}$$

$$A_3 = \int_0^{L_p} \frac{L^2 \, dL}{(r_o - mL)^4} \tag{36}$$

$$A_4 = \int_0^{L_p} \frac{L^3 \, dL}{(r_o - mL)^4} \tag{37}$$

$$A_5 = \int_0^{L_p} \frac{L^4 \, dL}{(r_o - mL)^4} \tag{38}$$

$$A_6 = \int_0^{L_p} \frac{L^5 \, dL}{(r_o - mL)^4} \tag{39}$$